STABILITY OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES IN THE CRITICAL CASE

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In the papers [1 to 4] there was found a linear approximation for a system of differential equations $z^\cdot = f(z,t)$ with discontinuous right-hand sides, and there were proved certain theorems on the stability of continuous solutions of this system. Below we consider the stability of periodic solutions of this system in critical cases.

Let the system of differential equations be given in vector form

$$z^\cdot = f(z, t) \qquad (0.1)$$

The function $f(x,t)$ is given in an $n+1$ dimensional curvilinear cylinder C whose axis is a continuous integral curve $z = z^\circ(t)$ of the system (0.1). Furthermore, the function $f(z,t)$ is periodic of period τ .

The hypersurfaces (the surfaces of discontinuity)

$$F_\alpha(z, t) = 0, \qquad [F_\alpha(z, t+\tau) = F_\alpha(z, t)] \qquad (0.2)$$

divide the cylinder C into regions H_α and intersect the curve $z = z^\circ(t)$ at the points M_α when $t = t_\alpha$. The right-hand sides of the system (0.1) satisfy the following conditions.

1. The functions $f_i(z,t)$ are continuous in each of the regions H_α (including their boundaries) and are continuously differentiable up to the order N , while during their passage through the surfaces (0.2), they and all their partial derivatives up to the Nth order, have discontinuities of the first kind only.

2. In the angular regions between the surfaces (0.2) and the planes $t = t_\alpha$, the following conditions are satisfied:

$$\frac{\partial^m f_i(z, t)}{\partial t^{m_0} \partial z_1^{m_1} \ldots \partial z_n^{m_n}} - \frac{\partial^m f_i(z^\circ, t)}{\partial t^{m_0} \partial z_1^{m_1} \ldots \partial z_n^{m_n}} \to \pm \xi_i^{m_0 m_1 \ldots m_n}$$
$$f_i(z, t) - f_i(z^\circ, t) \to \pm \xi_i \quad \text{for} \quad (z, t) \to M_\alpha \qquad (0.3)$$

The signs, plus and minus, correspond to the values $t = t_\alpha + 0$ and $t = t_\alpha - 0$.

3. The surfaces (0.2) are continuous, and at the points M_α they are smooth up to the order N , and along the integral curve $z = z^\circ(t)$

$$(F_\alpha^{\cdot})_{M_\alpha} \neq 0 \; (F_\alpha^{\cdot})^+_{M_\alpha} / (F_\alpha^{\cdot})^-_{M_\alpha} \geqslant \Gamma > 0 \qquad \left(F_\alpha^{\cdot} = \left[\sum_{i=1}^{n} \frac{\partial F_\alpha}{\partial z_i} f_i + \frac{\partial F_\alpha}{\partial t}\right]_{z=z^\circ(t)}\right) \quad (0.4)$$

Let us introduce the variable $x = z - z^\circ(t)$. The system (0.1) takes on the form

$$x^{\cdot} = p(x, t), \qquad p(x, t) = f(z^\circ + x, t) - f(z^\circ, t) \quad (0.5)$$

and the equations of the surfaces of discontinuity become

$$\Phi_\alpha(x, t) = 0, \qquad \Phi_\alpha(x, t) \equiv F_\alpha(z^\circ + x, t) \quad (0.6)$$

It is obvious that the stability of the solution $z = z^\circ(t) \equiv z^\circ(t + \tau)$ of the problem (0.1) is equivalent to the null solution of the system (0.5). The surfaces (0.6), in contrast to (0.2), will not be smooth at the points M_α, but will have breaks. At the intersections of regions H_α and $t_\alpha \leqslant t \leqslant t_{\alpha+1}$ (in the sequel we shall call these regions the central regions) the system (0.1) can be written in the form (*)

$$z_i^{\cdot} = f_i(z^\circ, t) + \sum \frac{1}{m!} \frac{\partial^m f_i(z^\circ, t)}{\partial z_1^{m_1} \ldots \partial z_n^{m_n}} (z_1 - z_1^\circ)^{m_1} \ldots (z_n - z_n^\circ)^{m_n} + R_i(z, t) \quad (0.7)$$

The partial derivatives occurring in Formula (0.7) are contunuous in each interval $t_\alpha \leqslant t \leqslant t_{\alpha+1}$, the functions $R_i(z,t)$ satisfy the conditions (0.8)

$$|R_i(z, t)| < a|x|^{N+1}, \qquad |x| = \sqrt{(z_1 - z_1^\circ)^2 + \ldots + (z_n - z_n^\circ)^2} \qquad (a = \text{const} > 0)$$

The surfaces of discontinuities can be written, to within infinitesimals of order higher than N , in the form

$$t_\alpha - t = \sum h_\alpha^{m_1 \ldots m_n} x_1^{m_1} \ldots x_n^{m_n} \quad (0.9)$$

(in the neighborhood of the point M_α of the lowest angular region).

We shall seek an approximate solution of Equation (0.5) in the neighborhood of the point $P(\zeta, t_1)$ which lies on the surface of discontinuities in the neighborhood of the point M_α in the lower angular region

$$x_i = \zeta_i + \sum_{m=0}^{N-1} \frac{(t - t_1)^{m+1}}{(m+1)!} \frac{d^m p_i(\zeta, t_1)}{dt^m} + p_i^*(\zeta, t), \quad \frac{dp_i}{dt} = \sum_{j=1}^{n} \frac{\partial p_i}{\partial x_j} p_j + \frac{\partial p_i^*}{\partial t} \quad (0.10)$$

The point $P(\zeta, t_1)$ lies on the surface of discontinuities. Therefore,

$$t_\alpha - t_1 = \sum h_\alpha^{m_1 \ldots m_n} \zeta_1^{m_1} \ldots \zeta_n^{m_n} = G(\zeta) \quad (0.11)$$

to within infinitesimals of order higher than N .

Now, if we assume $t = t_\alpha$ in (0.10), and if we replace $t_\alpha - t_1$ by its expression in (0.11), we obtain

$$x_i = \zeta_i + \sum_{m=0}^{N-1} \frac{G^{m+1}(\zeta)}{(m+1)!} \frac{d^m}{dt^m} p_i[\zeta, t_\alpha - G(\zeta)] + p_i^*(\zeta, t_\alpha) \quad (0.12)$$

We expand the last expression in increasing powers of $\zeta_1, \ldots, \zeta_n$, and we restrict the expansion to terms of order not greater than N relative to $\zeta_1, \ldots, \zeta_n$

$$x_i^+ = \sum b_i^{m_1 \ldots m_n} \zeta_1^{m_1} \ldots \zeta_n^{m_n} \quad (0.13)$$

The coefficients $b_j^{m_1 \ldots m_n}$ depend only on those $h_\alpha^{s_1 \ldots s_n}$ and $\xi_\alpha^{j_1 \ldots j_n}$, for which $s_1 + \ldots + s_n \leqslant m$, $j_1 + \ldots + j_n \leqslant m$ $(m = m_1 + \ldots + m_n)$. In particular, if $m_1 + \ldots + m_n = 1$, we obtain

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*) Here, and in what follows we assume, unless explicitely stated otherwise, that the indices of summation run through all possible values satisfying the inequalities $1 \leqslant m_1 + \ldots + m_n \leqslant N$.

$$b_i^{0\ldots010\ldots0} = b_{ij} = \delta_{ij} + \xi_i h_j \quad (\delta_{ij} \text{ is Kronecker's delta}) \tag{0.14}$$

Let us now consider the system of differential equations

$$x_i^{\cdot} = \sum \frac{1}{m!} \frac{\partial^m f_i(z^\circ, t)}{\partial z_1^{m_1} \ldots \partial z_n^{m_n}} x_1^{m_1} \ldots x_n^{m_n} \tag{0.15}$$

We shall look for a solution of this system in the neighborhood of the point $P(\zeta, t_1)$ which lies on the surface of discontinuities. Repeating the evaluations which were performed above, we obtain

$$x_i^- = \zeta_i + \sum c_i^{m_1\ldots m_n} \zeta_1^{m_1} \ldots \zeta_n^{m_n} \qquad (2 \leqslant m_1 + \ldots + m_n \leqslant N) \tag{0.16}$$

The coefficients $c_i^{m_1\cdots m_n}$ are evaluated by the same rules as the $b_i^{m_1\cdots m_n}$.

Equation (0.16) is inverted (solved) to within infinitesimals of order not higher than N

$$\zeta_i = \sum d_i^{m_1\ldots m_n} (x_1^-)^{m_1} \ldots (x_n^-)^{m_n} \tag{0.17}$$

Now, if we substitute ζ_i from (0.17) into (0.13) and if we limit ourselves to infinitesimals of order not higher than N, then (0.13) takes on the form

$$x_i^+ = \sum l_i^{m_1\ldots m_n} (x_1^-)^{m_1} \ldots (x_n^-)^{m_n} \quad \text{or} \quad x_\alpha^+ = S_\alpha(x_\alpha^-) \tag{0.18}$$

The system (0.15), together with the conditions of discontinuities (0.18), we will call a system of first approximation of the Nth order (*).

In particular, for $N = 1$, the system (0.15), (0.18) will take on the form

$$x_i^{\cdot} = \sum_{j=1}^{n} \frac{\partial f_i(z^\circ, t)}{\partial z_j} x_j, \qquad x_i^+ = x_i^- + \sum_{j=1}^{n} \xi_i h_j x_j^- \tag{0.19}$$

because of (0.14).

This system of the first approximation was considered in [2]. The system (0.16),(0.18) will, from now on, play a fundamental role.

1. If we let $X(t)$ denote the fundamental matix of the system (0.19), then we shall have Equation

$$X(t + \tau) = X(t)U \tag{1.1}$$

where U is a constant nonsingular matirx. Let us apply the transformation

$$x = L(t)y, \qquad L(t) = X(t)e^{-At}, \qquad A = \frac{1}{\tau} \ln U$$

to (0.5).

The system (0.5) will take on the form

$$y^{\cdot} = q(y, t), \qquad q(y, t) = L^{-1}p - L^{-1}L^{\cdot} y \tag{1.2}$$

If one takes the matix A in the Jordan form, then the system (1.2) will have the form

$$y_i^{\cdot} = \lambda_i y_i + \alpha_{i-1} y_{i-1} + \sum_{m=2}^{N} Y_i^{(m)}(y, t) + R_i^*(y, t) \tag{1.3}$$

in central regions.

Here, $|R_i^*(y, t)| < a_1 |y|^{N+1}$, $Y_i^{(m)}(y, t)$ are forms of order m in the variables $y_1, \ldots, y_n$ with periodic coefficients which are discontinuous at $t = t_\alpha$.

Let us transform the system (1.3) with the aid of the nonlinear transformation to a form in which the terms of order less or equal to N have constant and everywhere equal coefficients.

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*) It is not difficult to prove that Equations (0.18) will hold for the lower as well as for the upper angular regions.

Such a transformation can be performed if the characteristic numbers of the matrix A and the period τ are not connected by any relations of the form

$$m_1\lambda_1 + \ldots + m_n\lambda_n - \lambda_s = \pm 2\pi i\tau^{-1} \tag{1.4}$$

where $m_1, \ldots m_n$ are nonnegative integers.

Let us assume that these conditions are fulfilled. We shall seek a transformation in the form

$$y_i = u_i + \sum A_i^{m_1\ldots m_n}(t)\, u_1^{m_1} \ldots u_n^{m_n} \qquad (2 \leqslant m_1 + \ldots + m_n \leqslant N) \tag{1.5}$$

where the $A_i^{m_1\ldots m_n}(t)$ are periodic functions of period τ which are discontinuous at $t = t_\alpha$

In the new variables the system (1.3) takes on the form

$$u_i^{\cdot} = \lambda_i u_i + \alpha_{i-1}u_{i-1} + \sum a_i^{m_1\ldots m_n} u_1^{m_1} \ldots u_n^{m_n} + U_i(u, t)$$
$$(2 \leqslant m_1 + \ldots + m_n \leqslant N) \tag{1.6}$$

The conditions on the discontinuities for the system of the first approximation of order N of the system (1.3) has the form

$$y_i^+ = y_i^- + \sum g_i^{m_1\ldots m_n} (y_1^-)^{m_1} \ldots (y_n^-)^{m_n} \tag{1.7}$$

We will determine the discontinuities of the functions $A_i^{m_1\ldots m_n}(t)$ so that the conditions of discontinuities of the system of first approximation of the Nth order (1.6) may have the form

$$u_i^+ = u^-_i \tag{1.8}$$

to within terms of order higher than N .

Let us show that this can be done. Indeed, Equation

$$\sum A_i^{m_1\ldots m_n}(t_\alpha + 0)\, u_1^{m_1} \ldots u_n^{m_n} = \sum A_i^{m_1\ldots m_n}(t_\alpha - 0)\, u_1^{m_1} \ldots u_n^{m_n} +$$
$$+ \sum g_i^{m_1\ldots m_n}\Big[u_1 + \sum A_1^{s_1\ldots s_n}(t_\alpha - 0)\, u_1^{s_1} \ldots u_n^{s_n}\Big]^{m_1} \ldots$$
$$\ldots \Big[u_n + \sum A_n^{s_1\ldots s_n}(t_\alpha - 0)\, u_1^{s_1} \ldots u_n^{s_n}\Big] \tag{1.9}$$

must be satisfied up to within terms of order higher than N

Equating the coefficients of like powers of $u_1 \ldots, u_n$, we obtain the required conditions on the discontinuities. In Equation (1.9) the quantities $A_i^{m_1\ldots m_n}(t_\alpha - 0)$ are assumed to be known; therefore $A_i^{m_1\ldots m_n}(t_\alpha + 0)$ is uniquely determined.

Equation (1.9), just as (1.8), must be taken not in the exact sense, but with an accuracy up to infinitesimals of order higher than N .

In order to be able to reduce the system (1.3) to the form (1.6), it is necessary [5] that the coefficients $A_i^{m_1\ldots m_n}(t)$ of the transformation (1.5) satisfy Equations

$$\frac{d}{dt} A_i^{m_1\ldots m_n} + \Big(\sum_{s=1}^{n} m_s\lambda_s - \lambda_i\Big) A_i^{m_1\ldots m_n} = - a_i^{m_1\ldots m_n} + B_i^{m_1\ldots m_n}(t) \tag{1.10}$$

Here, $B_i^{m_1\ldots m_n}(t)$ are linear functions of the already known quantities $A_s^{k_1\ldots k_n}(t)$ with periodic (of period τ) coefficients which are discontinuous at $t = t_\alpha$. We shall seek a periodic solution of the system (1.10), which has discontinuities at $t = t_\alpha$ determined by (1.9).

Let us assume that all coefficients of the transformation (1.5) which appear in $B_i^{m_1\ldots m_n}(t)$ are periodic. We will show that then the $A_i^{m_1\ldots m_n}(t)$ can be selected so that they are periodic. Indeed, if $m_1\lambda_1 + \ldots + m_n\lambda_n - \lambda_i = 0$, then, in order that the $A_i^{m_1\ldots m_n}(t)$ be periodic, it is necessary and sufficient that

$$a_i^{m_1\ldots m_n} = \frac{1}{\tau}\int_0^\tau B_i^{m_1\ldots m_n}(t)\,dt + \sum_{\alpha=1}^h \Delta_\alpha A_i^{m_1\ldots m_n}$$
$$(\Delta_\alpha A_i^{m_1\ldots m_n} = A_i^{m_1\ldots m_n}(t_\alpha+0) - A_i^{m_1\ldots m_n}(t_\alpha-0)) \qquad (1.11)$$

where h is the number of surfaces of discontinuities.

All integrals exist because the discontinuities of the integrands are of the first order. If, however, $m_1\lambda_1 + \ldots + m_n\lambda_n - \lambda_i \neq 0$, then Equation (1.10) has a periodic solution for an arbitrary constant $a_i^{m_1\cdots m_n}$. Therefore, one can set $a_i^{m_1\cdots m_n} = 0$.

Thus, one can determine successively the coefficients of the transformation (1.5). This transformation is such that stability with respect to the variables y_s is equivalent to stability with respect to the variables u_j.

It is also obvious that the following estimate holds for the terms $U_i(u,t)$ of the system (1.6)

$$|U_i(u,t)| < a_2|u|^{N+1} \qquad (|u| = \sqrt{|u_1|^2 + \ldots + |u_n|^2}) \qquad (1.12)$$

2. Let us now go over to the investigation of the stability of the solution $z = z^\circ(t)$. Suppose that l of the characteristic numbers of the matrix A are equal to zero, and $2r$ of the characteristic numbers are pure imaginary

$$\lambda_j = 0 \quad (j = 1,\ldots,l), \qquad \lambda_{k+l} = \omega_k\sqrt{-1}, \qquad \lambda_{k+l+r} = -\omega_k\sqrt{-1} \quad (k = 1,\ldots,r)$$

Let us assume that there exists no relation of the form

$$m_1\omega_1 + \ldots + m_r\omega_r = 0 \quad \text{for arbitrary integers } m_1,\ldots,m_r \qquad (2.1)$$

between the characteristic frequencies.

Furthermore, let us assume that to the l null values of the characteristic numbers of the matix A there correspond simple elementary divisors. The rest of the characteristic numbers of the matrix A have negative real parts (actually, if only one of the characteristic numbers of the matrix A is positive, then the solution $z = z^\circ(t)$ will be unstable, as was shown in [2]). The peculiarity of the system (1.6) (to which the system (0.1) can be reduced in the central regions) is the fact that the variables $u_{l+2r+1},\ldots,u_n$ of the stable part of the system enter into the critical part of the system only in the terms $U_i(u,t)$ $(i = 1,\ldots,l+2r)$.

Therefore, one can write the critical part of (1.6) more precisely in the form

$$u_i^{\cdot} = \lambda_i u_i + \sum a_i^{m_1\ldots m_{l+2r}0\ldots0} u_1^{m_1}\ldots u_{l+2r}^{m_{l+2r}} + U_i(u,t)$$
$$(i = 1,2,\ldots,l+2r;\ 2 \leqslant m_1 + \ldots + m_{l+2r} \leqslant N) \qquad (2.2)$$

Now, we replace (in the system (2.2)) the variables which correspond to the pure imaginary characteristic numbers of the matrix A by

$$u_{l+k} = \rho_k e^{i\vartheta_k}, \qquad u_{l+r+k} = \rho_k e^{-i\vartheta_k} \qquad (2.3)$$

Then the system (2.2) takes the form

$$u_j^{\cdot} = \sum_{m=\nu}^N V_j^{(m)}(u_1,\ldots,u_l,\rho_1,\ldots,\rho_r) + V_j(u,\rho,\vartheta,t)$$
$$\rho_k^{\cdot} = \sum_{m=\nu}^N R_k^{(m)}(u_1,\ldots,u_l,\rho_1,\ldots,\rho_r) + R_k(u,\rho,\vartheta,t) \qquad (2.4)$$
$$\vartheta_k^{\cdot} = \sum_{m=2}^N \theta_k^{(m)}(u_1\ldots u_l,\rho_1,\ldots,\rho_r) + \theta_k(u,\rho,\vartheta,t) + \omega_k$$

Here $V_j^{(m)}(u,\rho)$, $R_k^{(m)}(u,\rho)$ and $\theta_k^{(m)}(u,\rho)$ are mth order forms with constant real coefficients (because the initial system (0.1) was real).

After the transformation (2.3), the stable part of the system takes on the form

$$u_i^{\cdot} = \lambda_i u_i + \alpha_{i-1} u_{i-1} + \sum_{m=2}^{N} U_k^{(m)}(u, \rho, \vartheta) + U_k(u, \rho, \vartheta, t) \qquad (2.5)$$

where $U_k^{(m)}(u, \rho, \vartheta)$ are mth order forms in the variables $u_1, \ldots, u_l, \rho_1, \ldots \ldots \rho_r, u_{l+2r+1}, \ldots, u_n$ with periodic coefficients which are continuous in ϑ.

As is known [5 and 6] one can, by changing only the stable variables by means of Formulas

$$u_i = \zeta_i + \varphi_i(u_1, \ldots, u_l, \rho_1, \ldots, \rho_r, \vartheta_1 \ldots \vartheta_r) \qquad (2.6)$$

insure that the expansion of the right-hand side of the stable part of the system of equations, for $\zeta_{l+2r+1} = \ldots = \zeta_n = 0$, contain terms of the critical part of the system to a degree exceeding N. After the application of the transformation (2.6), the system (2.5) takes on the form

$$\zeta_i^{\cdot} = \lambda_i \zeta_i + \alpha_{i-1}\zeta_{i-1} + Z_i(u, \rho, \vartheta, \zeta, t) \qquad (2.7)$$

where $Z_i(u, \rho, \vartheta, \zeta, t)$ satisfy the inequality

$$|Z_i(u, \rho, \vartheta, \zeta, t)| < \gamma(|\rho + u|^N + |\rho + u||\zeta| + |\zeta|^2) \qquad (\gamma = \text{const} > 0) \qquad (2.8)$$

for small $|u + \rho| + |\zeta|$.

In consequence of the transformation, the system (0.1), in the central regions, takes on the form

$$u_j^{\cdot} = V_j^{(\nu)}(u, \rho) + V_j(u, \rho, \vartheta, \zeta, t), \qquad \vartheta_k^{\cdot} = \omega_k + \theta_k(u, \rho, \vartheta, \zeta, t)$$
$$\rho_k^{\cdot} = R_k^{(\nu)}(u, \rho) + R_k(u, \rho, \vartheta, \zeta, t), \qquad \zeta_i^{\cdot} = \lambda_i\zeta_i + \alpha_{i-1}\zeta_{i-1} + Z_i(u, \rho, \vartheta, \zeta, t) \qquad (2.9)$$

Here $V_j(u, \rho, \vartheta, \zeta, t)$ and $R_k(u, \rho, \vartheta, \zeta, t)$ are periodic functions in ϑ and t, and they satisfy, for small enough $|u + \rho| + |\zeta|$, the inequalities

$$|R_j(u, \rho, \vartheta, \zeta, t)| < \alpha(|\rho + u|^{\nu+1} + |\zeta|^2) \qquad (\alpha = \text{const} > 0)$$
$$|V_j(u, \rho, \vartheta, \zeta, t)| < \beta_1(|\rho + u|^{\nu+1} + |\zeta|^2) \qquad (\beta_1 = \text{const} > 0) \qquad (2.10)$$

One can prove the following theorem.

T h e o r e m . If the null solution of the system

$$u_j^{\cdot} = V_j^{(\nu)}(u, \rho), \qquad \rho_k^{\cdot} = R_k^{(\nu)}(u, \rho) \qquad (2.11)$$

is asymptotically stable then the null solution of the system (2.9) is asymptotically stable. Indeed, because of the asymptotic stability of the null solution of the system (2.11), there exists [7] a continuously differentiable function $V_1(u, \rho)$ satisfying the following inequalities

$$b_1|u+\rho|^B \leqslant V_1(u, \rho) \leqslant b_2|u + \rho|^B, \quad \left|\frac{\partial V_1}{\partial u_j}\right| \leqslant b_4|u + \rho|^{B-1}$$
$$\left(\frac{dV_1}{dt}\right)_{(2.11)} \leqslant -b_3|u + \rho|^{B+\nu-1}, \qquad \left|\frac{\partial V_1}{\partial \rho_k}\right| \leqslant b_4|u + \rho|^{B-1} \qquad (2.12)$$

where B, b_1, b_2, b_3 and b_4 are positive constants.

Let us construct the function $V_2(\zeta)$ for the linear part of the system (2.7) which satisfies the inequalities

$$c_1|\zeta|^2 \leqslant V_2(\zeta) \leqslant c_2|\zeta|^2$$
$$\left(\frac{dV_2}{dt}\right)_0 = \sum_{s=l+2r+1}^{n} \frac{\partial V_2}{\partial \zeta_s}(\lambda_s\zeta_s + \alpha_{s-1}\zeta_{s-1}) \leqslant -c_3|\zeta|^2, \qquad \left|\frac{\partial V_2}{\partial \zeta_s}\right| \leqslant c_4|\zeta| \qquad (2.13)$$

Next we observe the changes of the function

$$V(u, \rho, \zeta) = V_1(u, \rho) + V_2(\zeta) \qquad (2.14)$$

along the discontinuous trajectories of the system (2.9) into which the continuous trajectories of (0.1) are transformed in the neighborhood of the solution $z = z^\circ(t)$.

In the central regions of the space $(u, \rho, \zeta, \vartheta, t)$, into which the central regions of the space (x,t) are mapped, the function $V(u,\rho,\zeta)$ satisfies the inequality

$$V(u, p, \zeta) \leqslant b_2 |u + \rho|^B + c_2 |\zeta|^2 \tag{2.15}$$

Let us now evaluate $V^\cdot$ with the aid of (2.9)

$$\left(\frac{dV}{dt}\right)_{(2.9)} = \left(\frac{dV_1}{dt}\right)_{(2.11)} + \left(\frac{dV_2}{dt}\right)_0 + \sum_{k=1}^{r} \frac{\partial V_1}{\partial \rho_k} R_k + \sum_{j=1}^{l} \frac{\partial V_1}{\partial u_j} V_j + \sum_{i=l+2r+1}^{n} \frac{\partial V_2}{\partial \zeta_i} Z_i \tag{2.16}$$

Because of the estimates (2.12),(2.13),(2.8) and (2.10) we obtain the inequalities

$$\left| \sum_{k=1}^{r} \frac{\partial V_1}{\partial \rho_k} R_k + \sum_{j=1}^{l} \frac{\partial V_1}{\partial u_j} V_j \right| \leqslant b_4 (\alpha + \beta_1) | u + \rho |^{B-1} (| u + \rho |^{\nu+1} + | \zeta |^2) \tag{2.17}$$

$$\left| \sum_{i=l+2r+1}^{n} \frac{\partial V_2}{\partial \zeta_i} Z_i \right| \leqslant c_4 | \zeta | \gamma (| \rho + u |^N + | u + \rho | (| \zeta | + | \zeta |^2) \tag{2.18}$$

If $N \geqslant B + \nu$ and $| u + \rho | + |\zeta|$ are small enough, then we obtain, for (2.16), the estimate

$$\left(\frac{dV}{dt}\right)_{(2.13)} < -d_1 |u + \rho|^{B+\nu} - d_2 |\zeta|^2 < -\mu^2 (|u + \rho| + |\zeta|)^{B+\nu} \tag{2.19}$$

where d_1 and d_2 are positive constants.

Since the sonstant B in the inequalities (2.12) is sufficiently large [7], it follows from (2.15) that

$$V(u, \rho, \zeta) < \mu_1^2 (| u + \rho | + | \zeta |)^2 \tag{2.20}$$

Comparing the inequalities (2.19) and (2.20), we obtain

$$\frac{dV}{dt} < -\mu^2 (\mu_1^{-2} V)^{1/2(B+\nu)} \tag{2.21}$$

$$V < V_0 [1 + V_0^{N_1} \beta (t - t_0)]^{-1/N_1}, \qquad \beta = N_1 \mu^2 \mu_1^{-2(N_1+1)}, \qquad N_1 = 1/2 (B + \nu + 1) \tag{2.22}$$

Let us now watch the changes of $V(u,\rho,\zeta)$ in the angular regions. Because the null solution of the system (2.11) is by hypothesis asymptotically stable, it will also be stable for the system

$$u_j^\cdot = \sum_{m=\nu}^{N} V_j^{(m)}(u, \rho), \quad \rho_k^\cdot = \sum_{m=\nu}^{N} R_k^{(m)}(u, \rho) \qquad (j = 1, \ldots, l;\ k = 1, \ldots, r)$$

$$\zeta_i^\cdot = \lambda_i \zeta_i + \alpha_{i-1} + \zeta_{i-1} + \sum_{m=2}^{N} Z_i^{(m)}(u, \rho, \vartheta, \zeta) \qquad (i = l + 2r + 1, \ldots, n)$$

$$N > (N_1 + 1) B - 1 \tag{2.23}$$

To the continuous solution of the system (2.23) there corresponds a discontinuous solution of the system (0.16),(0.18) of the first approximation of the Nth order.

If we denote by η the vector $\eta = u + \rho + \zeta$, then $V(\eta^+) = V(\eta^-)$ when $\eta^+ = \eta^-$.

Thus,

$$V(x^+, t_\alpha) = V(x^-, t_\alpha) + o(| x |^{N+1}) \quad \text{for} \quad x^+ = S(x^-) + o(| x |^N) \tag{2.24}$$

Next, let us denote by $M(x_1, t_1)$ the point of intersection of a trajectory of the system (0.5) with the surface of discontinuities $\Phi_\alpha(x, t) = 0$, Then by the construction of the system of the first approximation

$$x = x^+ + o(|x|^N) \tag{2.25}$$

where x is a point on the trajectory of the system (0.5) for $t = t_\alpha$

From the inequalities (2.24) and (2.25) it follows that

$$V(x, t_\alpha) = V(x^-, t_\alpha) + o(|x|^{N+1}) \tag{2.26}$$

This implies directly that

$$-\varkappa < V^{-N_1}(x^-, t_\alpha) - V^{-N_1}(x, t_\alpha) < \varkappa \tag{2.27}$$

where $\varkappa$ is an arbitrary small positive number.

Let us select a cylinder of radius δ so small that the time which the trajectory stays in any angular region be sufficiently small and that

$$\varkappa < \beta T \qquad (T = \min|t_{\alpha+1} - t_\alpha|) \tag{2.28}$$

The radius of the cylinder can be chosen so small that the planes $t = t_\alpha^* = {}^1/_2\,[t_{\alpha+1} + t_\alpha]$ do not intersect the angular regions inside the cylinder C.

The coefficient of growth of $V(x,t)$ is finite [2] in the angular regions. Let us denote the largest coefficient of growth by Q. We take the initial point $x_1^* = x(t_1^*)$ inside the cylinder $V = \delta_1 = \delta/Q$.

As t changes within the bounds $t_1^* \leqslant t \leqslant t_2^*$, the function $V(x,t)$, taken along the integral curve of the system (0.5), decreases in the central region to the point (x_1, t_1) which is the intersection of the curve with the surface of discontinuity. Next, in the angular region, it can increase, but this increase will be compensated by jumps of the function for $t = t_\alpha$.

Indeed, because of (2.22)

$$V(x_1, t_1) < V_1[1 + V_1^{N_1}\beta(t_1 - t_1^*)]^{-1/N_1}, \qquad V_1 = V(x_1^*, t_1^*) \tag{2.29}$$

Furthermore, (2.30)

$$V(x^-, t_\alpha) < V(x_1, t_1)[1 + V^{N_1}(x_1, t_1)\beta(t_\alpha - t_1)]^{-1/N_1} < V_1[1 + V_1^{N_1}\beta(t_\alpha - t_1^*)]^{-1/N_1}$$

Because of (2.27)

$$V(x, t_\alpha) < V(x^-, t_\alpha)[1 - \varkappa V^{N_1}(x^-, t_\alpha)]^{-1/N_1} \tag{2.31}$$

Using (2.31) and (2.30), we obtain

$$V(x, t_\alpha) < V_1\{1 + V_1^{N_1}[\beta(t_\alpha - t_1^*) - \varkappa]\}^{-1/N_1}$$

In the next central region the function $V(x,t)$ will decrease according to (2.29), $V_2 = V(x_2^*, t_2^*) < V_1\{1 + V_1^{N_1}[\beta(t_2^* - t_1^*) - \varkappa]\}^{-1/N_1} < V_1$

since by hypothesis $\varkappa$ satisfies the inequality (2.28).

If one takes into consideration that the coefficient of growth of $V(x,t)$ in the angular regions does not exceed Q, one may conclude that the integral curve, for $t_1^* \leqslant t \leqslant t_2^*$ passes through the cylinder $V = \delta$. Repeating the preceding argument for the following regions with $t_2^* \leqslant t \leqslant t_3^*, \ldots$ $\ldots, t_i^* \leqslant t \leqslant t^*_{i+1}$ and so on, we obtain

$$V_{i+1} < V_i\,\{1 + V_i^{N_1}[\beta(t_{i+1}^* - t_i^*) - \varkappa]\}^{-1/N_1} < V_i[1 + V_i^{N_1}(\beta T - \varkappa)]^{-1/N_1} < V_i \tag{2.32}$$

Because of the inequality (2.32), the sequence $\{V_i\}$ has a limit.

Let us suppose that this limit is not zero, i.e. $\lim V_i > 0$ as $t \to \infty$. Then V must satisfy the inequality

$$V \leqslant V[1 + V^{N_1}(\beta T - \varkappa)]^{-1/N_1} \qquad \text{or} \qquad V \leqslant 0$$

Thus, it is proved that $\lim V_i = 0$ as $t \to \infty$; this implies the asymptotic stability of the solution $z = z^\circ(t)$ of the system (0.1).

The theorem asserts furthermore, that to answer the question on the stability, one must seek an approximation of order ν, because the coefficients of the system (2.11) depend only on the coefficients of an approximation of order ν.

In conclusion let us consider the case when the matrix A has one characteristic number equal to zero, while the remaining characteristic numbers have negative real parts. In this case the system (2.11) will consist of one equation

$$u_1^{\cdot} = g u_1^{\nu}$$

Then the true theorem reads as follows: if ν is an odd number, and $g < 0$, then the solution $z = z^0(t)$ of the system (0.1) is asymptotically stable.

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